The Sprinkling Problem*

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ABSTRACT

Let W be a matrix-valued linear function of a vector variable x with the property that each entry of $W(x)$ is either 0 or an entry of x. Call such a function a sprinkling. Letting $N(\cdot)$ denote the spectral norm on matrices and $n(\cdot)$ the Euclidean norm on vectors, the problem is to characterize those sprinklings W such that $N(W(x)) \leqslant n(x)$, in terms of the pattern of the entries of x in $W(x)$. We mention several equivalent formulations of this problem that motivated us, and describe some sufficient conditions on W for the inequality and some necessary conditions in terms of forbidden subpatterns in W. A complete solution remains open.

0. INTRODUCTION

Let $\mathbb{R}^t$ denote the vector space of real t-tuples, and $M_{r,d}$ the vector space of real r-by-d matrices. We call a linear mapping

$$W : \mathbb{R}^t \to M_{r,d}$$

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a *sprinkling* if each entry of $W(x)$ is either zero or an entry of the vector x. Letting $N(\cdot)$ denote the spectral norm and $n(\cdot)$ the Euclidean norm, the *sprinkling problem* is then to characterize those sprinklings W such that

$$N(W(x)) \leqslant n(x) \qquad \text{for all} \quad x \in \mathbb{R}^t. \tag{0.1}$$

As $W(x)$ is simply an arrangement (possibly with repeats or omissions) of the symbols from the list x in an r-by-d array, we view the sprinkling problem as a combinatorial one about the pattern of nonzero symbols in $W(x)$. After describing in the next section several equivalent formulations of the sprinkling problem that motivated us, we give a variety of sufficient conditions for the inequality (0.1) in the following two sections. We then give necessary conditions for the inequality in terms of forbidden subpatterns in $W(x)$. A complete solution of the sprinkling problem remains an intriguing open question.

1. ALTERNATIVE STATEMENTS OF THE PROBLEM

In this section, we state three different formulations of the sprinkling problem.

Let $W : \mathbb{R}^t \to M_{r,d}$ be a sprinkling. Define $p : \mathbb{R}^t \times \mathbb{R}^d \to \mathbb{R}^r$ by

$$p(x, y) = W(x)y. \tag{1.1}$$

Each entry of $p(x, y)$ is of the form

$$x^T P_i y$$

for a certain zero-one matrix P_i. Conversely, given a mapping p of this form, (1.1) defines a mapping W. Since the spectral norm is the operator norm induced by the Euclidean norm, p satisfies

$$n(p(x, y)) \leqslant n(x)n(y) \qquad \forall x, y$$

if and only if W is a sprinkling that satisfies the inequality (0.1). Questions about such objects arose in the second author's Ph.D. thesis [4] dealing with submultiplicativity of classes of matrix products in various norms.

We may also associate a sprinkling W with a three-dimensional zero-one array $A = [a_{ijk}]$. Define

$$a_{ijk} = \begin{cases} 1 & \text{if the } i, j \text{ entry of } W(x) \text{ is } x_k, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

Conversely, given a zero-one three-dimensional array

$$A = [a_{ijk}]$$

of size r by d by t, we may define a mapping $W : \mathbb{R}^t \to M_{r,d}$ by

$$(W(x))_{ij} = \sum_{k=1}^{t} a_{ijk} x_k.$$

The three-dimensional array A satisfies the inequality

$$\sum_{i=1}^{r} \sum_{j=1}^{d} \sum_{k=1}^{t} a_{ijk} x_i y_j z_k \leqslant n(x)n(y)n(z) \tag{1.3}$$

for all $x \in \mathbb{R}^r$, $y \in \mathbb{R}^d$, and $z \in \mathbb{R}^t$ if and only if W is a sprinkling that satisfies the inequality (0.1). This view reveals a duality among rows, columns, and variables in the original version of the sprinkling problem, which we exploit later.

Finally, given a sprinkling W, we may also write

$$W(x) = x_1 Q_1 + x_2 Q_2 + \cdots + x_t Q_t, \tag{1.4}$$

in which each Q_i is an r-by-d zero-one matrix. Conversely, given a set of t r-by-d zero-one matrices, (1.4) defines a mapping $W : \mathbb{R}^t \to M_{r,d}$. The inequality (0.1) is equivalent to a weighted-sum property that $\{Q_1, \cdots, Q_t\}$ might possess:

$$\sum_{i=1}^{t} |x_i|^2 \leqslant 1 \quad \text{implies} \quad N\left(\sum_{i=1}^{t} x_i Q_i\right) \leqslant 1.$$

2. ISOMETRIC EXTENSION OF SPRINKLINGS

Our purpose in this section is to give sufficient conditions for the inequality (0.1), all based upon a notion of extendability for a sprinkling.

Our starting point is the simple observation that $A \in M_{r,d}$ is a contraction in the spectral norm $[N(A) \leqslant 1]$ if and only if there is a $B \in M_{k,d}$ such that the matrix

$$\begin{bmatrix} A \\ B \end{bmatrix} \in M_{r+k,d}$$

is an isometry (i.e. has orthonormal columns). The sufficiency of the condition is clear, as extraction of submatrices cannot increase the spectral norm, and the necessity follows from letting B be any matrix such that B^TB is a factorization of the positive semidefinite matrix $I - A^TA$, so that

$$I = A^TA + B^TB = \begin{bmatrix} A \\ B \end{bmatrix}^T \begin{bmatrix} A \\ B \end{bmatrix}.$$

Clearly, the isometry condition may be changed to orthogonal columns of length at most one, without loss of generality. By restricting the domain of the sprinkling W to vectors x of Euclidean length one, we then have (due to homogeneity):

PROPOSITION 2.1. *The sprinkling $W : \mathbb{R}^t \to M_{r,d}$ satisfies the inequality (0.1) if and only if there is a mapping $V : \mathbb{R}^t \to M_{k,d}$ such that for every $x \in \mathbb{R}^t$ of Euclidean length one,*

$$\begin{bmatrix} W(x) \\ V(x) \end{bmatrix}$$

has orthogonal columns of Euclidean length at most one.

The matrix V of Proposition 2.1 might be of a particularly simple form (since W is). Likely candidates are the *signed sprinklings*: linear mappings $V : \mathbb{R}^t \to M_{k,d}$ such that entries of $V(x)$ are either 0, an entry of x, or an entry of $-x$; for example, if $W : \mathbb{R}^3 \to M_{3,2}$ is the sprinkling

$$W\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \\ b & c \end{bmatrix},$$

then $V : \mathbb{R}^3 \to M_{1,2}$, defined by

$$V\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{bmatrix} -c & b \end{bmatrix}$$

is a signed sprinkling realizing the condition of Proposition 2.1.

DEFINITION 2.2. A sprinkling $W : \mathbb{R}^t \to M_{r,d}$ is called *extendable* if there is a signed sprinkling $V : \mathbb{R}^t \to M_{k,d}$ such that

$$\begin{bmatrix} W(x) \\ V(x) \end{bmatrix}$$

has orthogonal columns of Euclidean length at most one for all $x \in \mathbb{R}^t$ such that $n(x) = 1$.

By Proposition 2.1, an extendable sprinkling satisfies the inequality (0.1). The converse is, unfortunately, false, as the sprinkling

$$W\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{bmatrix} b & a & 0 & 0 & 0 \\ 0 & c & b & 0 & 0 \\ 0 & 0 & d & c & 0 \\ 0 & 0 & 0 & e & d \\ e & 0 & 0 & 0 & a \end{bmatrix} \tag{2.1}$$

is not extendable. We postpone justification of this claim until after Proposition 2.4. That W does satisfy (0.1) follows from the fact the W^T is extendable. This may be seen in the straightforward way in which Proposition 2.1 and the idea of extendability are most naturally used: add rows with $+$ or $-$ chosen symbols in x (and zeros) so as to make pairs of columns formally orthogonal in turn. If this can be carried to completion without any column containing a repeated symbol ($+$ or $-$), then the sprinkling is extendable and satisfies (0.1), as no column can have Euclidean length exceeding that of x.

In the case of W^T, such an orthogonal extension is

$$\begin{bmatrix} b & 0 & 0 & 0 & e \\ a & c & 0 & 0 & 0 \\ 0 & b & d & 0 & 0 \\ 0 & 0 & c & e & 0 \\ 0 & 0 & 0 & d & a \\ c & -a & 0 & 0 & 0 \\ -e & 0 & 0 & 0 & b \\ 0 & d & -b & 0 & 0 \\ 0 & 0 & e & -c & 0 \\ 0 & 0 & 0 & a & -d \end{bmatrix}$$

The first five rows are those of W^T. The sixth row is added to make columns 1 and 2 orthogonal, the seventh to make columns 1 and 5 orthogonal, the eighth to make columns 2 and 3 orthogonal, the ninth to make columns 3 and 4 orthogonal, and the last to make columns 4 and 5 orthogonal. All other pairs of columns were already combinatorially orthogonal. As no column contains a repeated symbol at the completion of this process, this extension verifies that W^T, and thus W, satisfies (0.1).

Verification of extendability can be more subtle than the above example. (Rows with three or more nonzeros may be necessary.) However, the exact conditions under which W may be "fixed up" one pair of columns at a time, with rows of only two nonzero entries, is contained in the following proposition. This provides a more directly verifiable, though less general, sufficient condition for the inequality (0.1).

PROPOSITION 2.3. *The sprinkling W is extendable, and thus satisfies the inequality* (0.1), *if it satisfies the following criteria*:

(1) *each entry of x occurs at most once in each row of $W(x)$, and*

(2) *if the same entry of x occurs in two rows of $W(x)$, then those rows are combinatorially orthogonal* (*i.e. orthogonal for all vectors x*).

Proof. To each pair of nonzero entries in a row of $W(x)$, we assign a single row of $V(x)$, e.g., if $w_{ij} = x_p$ and $w_{ik} = x_q$, then there is a row, say the lth, of $V(x)$ that is all zero except $v_{lj} = x_q$ and $v_{lk} = -x_p$. Except for multiplying a row (or rows) of V by -1 and permuting rows of V, V is uniquely defined. Now, suppose that $\begin{bmatrix} W(x) \\ V(x) \end{bmatrix}$ has the same entry of x occurring more than once in a column, e.g., the symbol a occurs more than once. By permuting the rows of W and the columns of $\begin{bmatrix} W \\ V \end{bmatrix}$ we may obtain

$$\begin{bmatrix} W \\ V \end{bmatrix} = \begin{bmatrix} aI & W_{12} \\ W_{21} & W_{22} \\ V_1 & V_2 \end{bmatrix},$$

in which all the a's of W are contained in the upper left block. By hypothesis, the columns of W_{12} contain at most one entry each and none of these entries are a's. Therefore, our construction places at most one a in each column of V_2 and no a's in V_1, which contradicts the supposition that more than one a occurs in a column of $\begin{bmatrix} W(x) \\ V(x) \end{bmatrix}$. ■

We note that if criterion (2) of Proposition 2.3 does not hold, then W contains a submatrix of the form

$$\begin{bmatrix} a & 0 & b \\ 0 & a & c \end{bmatrix},$$

and then the naive construction of Proposition 2.3 yields

$$\begin{bmatrix} a & 0 & b \\ 0 & a & c \\ b & 0 & -a \\ 0 & c & -a \end{bmatrix},$$

so that this construction fails to provide an extension when the criteria are not satisfied. Note, however, that extension is still possible in this case:

$$\begin{bmatrix} a & 0 & b \\ 0 & a & c \\ b & c & -a \\ c & -b & 0 \end{bmatrix}.$$

EXAMPLE 2.4. As an indication of the information that Proposition 2.3 can convey easily, note that it implies that the largest singular value of the

$2r$-by-$2r$ matrix

$$W\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{bmatrix} a & b & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & c & d & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & c & d & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & a & b \\ d & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & c \end{bmatrix}$$

is no more than $(|a|^2 + |b|^2 + |c|^2 + |d|^2)^{1/2}$. This is because $W : \mathbb{R}^4 \to M_{2r,2r}$ is a sprinkling meeting the extendability condition in Proposition 2.3. We know of no other means by which this inequality is so obvious. Equality occurs for several vectors $(a, b, c, d)^T$, such as $(1, 1, 1, 1)$, $(1, 0, 0, 1)$, $(0, 1, 1, 0)$, $(1, 1, 0, 0)$, and $(0, 0, 1, 1)$.

It is clear that a sprinkling W satisfies the inequality (0.1) if and only if W^T does. Given a sprinkling $W : \mathbb{R}^t \to M_{r,d}$, we may define the mappings

$$W' : \mathbb{R}^d \to M_{r,t}$$

by

$$W'(z) = [Q_1 z, Q_2 z, \cdots, Q_t z],$$

and

$$W'' : \mathbb{R}^r \to M_{t,d}$$

by

$$W''(y) = \begin{bmatrix} y^T Q_1 \\ y^T Q_2 \\ \vdots \\ y^T Q_t \end{bmatrix},$$

in which $Q_1, \cdots, Q_t$ are defined by (1.4). Note the different ways that the matrices Q_i appear in W, W', and W''. The mappings W' (and W'') are sprinklings that satisfy (0.1) if and only if W is. This follows from the equality

$$y^T W(x) z = y^T W'(z) x = x^T W''(y) z,$$

which holds for all $x \in \mathbb{R}^t$, $y \in \mathbb{R}^r$, and $z \in \mathbb{R}^d$. We also note that with the correspondence $W \leftrightarrow [a_{ijk}]$ defined in (1.2), we have $W' \leftrightarrow [a_{ikj}]$ and $W'' \leftrightarrow [a_{kji}]$, just as $W^T \leftrightarrow [a_{jik}]$. Thus, we may decide if W satisfies (0.1) by considering any of W, W', W'' or their transposes. However, application of the following proposition shows that we need not construct W', W'' or their transposes to determine if one of them is extendable.

PROPOSITION 2.5. *Let W be a signed sprinkling. $W(x)$ has orthogonal columns, each of which contains only distinct entries of x, if and only if W'', defined above, has the following 2-by-2 block property: Every 2-by-2 submatrix of $W''(x)$ contains only distinct entries of x or is permutation equivalent to*

$$\pm\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

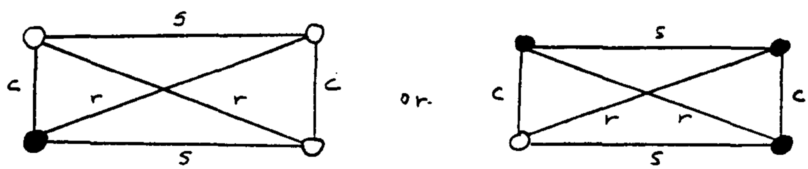
for distinct entries a, b of x.

Proof. Define a graph $G = (V, E)$ with the vertex set V the set of not identically zero entries of $W(x)$. Color the vertices corresponding to positively signed entries, x_k, white, and those corresponding to negatively signed entries, $-x_k$, black. The edge set E is partitioned into three subsets: E_r, E_c, and E_s. For distinct vertices v_i and v_j, $(v_i, v_j) \in E_r$ if the entries of $W(x)$ corresponding to v_i and v_j are in the same row, $(v_i, v_j) \in E_c$ if the corresponding entries are in the same column, and $(v_i, v_j) \in E_s$ if the corresponding entries are both $\pm x_k$ for some k, i.e., both entries arise from the same entry of x.

Suppose the i, j and i, k entries of $W(x)$ are both not identically zero. Then G contains an edge

$$\varnothing \overset{r}{\text{———}} \varnothing$$

(vertices either color). Since column j and column k are orthogonal for all x, this edge must occur in a subgraph of the form

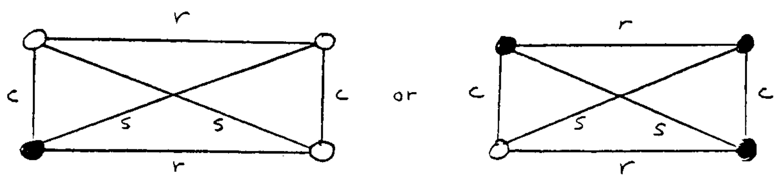


The transformation $W \to W''$ corresponds to $G \to G''$, with $V'' = V$, $E_s'' =$

E_r, $E_r'' = E_s$, and $E_c'' = E_c$. Thus, two entries w_{ij} and w_{pq} of W'' arise from the same entry of x if and only if the corresponding vertices in G'' are connected by an s-edge:

$$\varnothing \overset{s}{\text{———}} \varnothing$$

(vertices either color). This was formerly an r-edge in G; hence this edge is contained in a subgraph of the form



This means that the 2-by-2 submatrix in rows indexed by $\{i, p\}$ and columns indexed by $\{j, q\}$ is permutation equivalent to

$$\pm\begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$

in which a, b are distinct entries of x. This proves the forward implication; the reverse implication is obtained with the same technique. ■

Note that if W is a sprinkling, extending W is equivalent to finding a signed sprinkling U with the 2-by-2 block property such that

$$U\begin{pmatrix} x \\ 0 \end{pmatrix} = W''(x).$$

With W the sprinkling defined in (2.1),

$$W'' = \begin{bmatrix} 0 & a & 0 & 0 & e \\ a & 0 & b & 0 & 0 \\ 0 & b & 0 & c & 0 \\ 0 & 0 & c & 0 & d \\ e & 0 & 0 & d & 0 \end{bmatrix}.$$

There is no signed sprinkling U with the 2-by-2 block property such that

$$U\binom{x}{0} = W''(x).$$

Due to the 2-by-2 submatrices containing two a's, two b's, two c's, and two d's, the diagonal of U would have to be

$$\pm\begin{bmatrix} z & & & & \\ & -z & & & \\ & & z & & \\ & & & -z & \\ & & & & z \end{bmatrix}.$$

However, it is then impossible for the 2-by-2 submatrix containing the e's to have the 2-by-2 block property.

PROPOSITION 2.6. *Let W be a sprinkling. One of W, W', W'' or one of their transposes can be extended if and only if*

(1) *W can be extended, or*
(2) *W^T can be extended or*
(3) *there is a signed sprinkling U with the 2-by-2 block property such that*

$$W(x) = U\binom{x}{0}.$$

Proof. First, we note that

$$(W^T)'' = (W')^T$$

and

$$(W^T)' = (W')'' = (W'')^T.$$

Applying Proposition 2.5, we obtain W is extendable $\Leftrightarrow$ W'' satisfies (3). Then:

(a) W' is extendable $\Leftrightarrow$ $(W')'' = (W'')^T$ satisfies (3 $\Leftrightarrow$ W'' satisfies (3) $\Leftrightarrow$ W is extendable.
(b) W'' is extendable $\Leftrightarrow$ $(W'')'' = W$ satisfies (3).
(c) $(W')^T = (W^T)''$ is extendable $\Leftrightarrow$ $[(W^T)'']'' = W^T$ satisfies (3) $\Leftrightarrow$ W satisfies (3).

(d) $(W'')^T = (W')''$ is extendable $\Leftrightarrow$ $[(W')'']'' = W'$ satisfies (3) $\Leftrightarrow$ $(W')^T = (W^T)''$ satisfies (3) $\Leftrightarrow$ $[(W^T)'']'' = W^T$ is extendable. ■

We now state our most general sufficient condition for (0.1) based on extendability alone.

COROLLARY 2.7. *A sprinkling W satisfies the inequality* (0.1) *if* (1), (2), *or* (3) *of Proposition* 2.6 *holds.*

3. A COMPRESSION INEQUALITY

Given a sprinkling W, we can sometimes show that W satisfies (0.1) by considering several smaller sprinklings. For example, if W has the form of a direct sum,

$$W(x) = \begin{bmatrix} W_1(x) & 0 \\ 0 & W_2(x) \end{bmatrix},$$

the W satisfies (0.1) if both W_1 and W_2 do. We will state our most general proposition along these lines after introducing some notation and stating two lemmas. Let $\{U_1, \dots, U_p\}$, $\{V_1, \dots, V_q\}$, and $\{Y_1, \dots, Y_s\}$ be partition of $\{1, \dots, t\}$, $\{1, \dots, r\}$, and $\{1, \dots, d\}$, respectively. For $x \in \mathbb{R}^t$, let $x[U_i]$ denote the vector whose entries are the entries of x indexed by U_i. For a matrix $B \in M_{r,d}$, $B[V_i, Y_j]$ denotes the submatrix of B contained in rows indexed by V_i and columns indexed by Y_j.

LEMMA 3.1. *Let A, B be r-by-d matrices with nonnegative entries. If the inequality $A \leqslant B$ holds entrywise, then $N(A) \leqslant N(B)$.*

Proof. This follows from Theorems 1 and 2 in [2]. ■

LEMMA 3.2. *Let $A \in M_{r,d}$. Then*

$$N(A) \leqslant N\big(\big[N\big(A[V_i, Y_j]\big)\big]\big),$$

in which $[N(A[V_i, Y_j])]$ denotes the q-by-s matrix whose i, jth entry is $N(A[V_i, Y_j])$.

Proof. This is proven in [3]. ■

PROPOSITION 3.3. *Let $W : \mathbb{R}^t \to M_{r,d}$ be a sprinkling, and suppose that the system of partitions described above has the following property: For each i and j, $W(x)[V_i, Y_j]$ contains either no entries of x or entries of x all of whose indices appear in a single set U_k. Then W satisfies the inequality* (0.1) *provided*:

(1) *For each i, j, $N(W(x)[V_i, Y_j]) \leqslant n(x)$ for all $x \in \mathbb{R}^t$, i.e., $W[V_i, Y_j]$ satisfies* (0.1).

(2) $\hat{W}$ *defined by*

$$\hat{W} : \mathbb{R}^p \to M_{q,s}, \hat{W}(y) = \left[\hat{w}_{ij}\right],$$

$$\hat{w}_{ij} = \begin{cases} y_k & \text{if } W(x)\left[V_i, Y_j\right] \text{ contains entries of } x \text{ indexed by } U_k, \\ 0 & \text{if } W(x)\left[V_i, Y_j\right] = 0 \end{cases}$$

satisfies (0.1). *Note that the assumed property of the partitions insures that $\hat{w}_{ij}$ is well defined.*

Proof. Let $X \in \mathbb{R}^t$ be given. We claim that

$$N(W(x)) \leqslant N\left(\left[N\left(W(x)\left[V_i, Y_j\right]\right)\right]\right) \leqslant N\left(\hat{W}(\hat{x})\right) \leqslant n(\hat{x}) = n(x), \tag{3.1}$$

where

$$\hat{x} = \begin{bmatrix} n(x[U_1]) \\ n(x[U_2]) \\ \vdots \\ n\left(x\left[U_p\right]\right) \end{bmatrix}.$$

The first inequality in (3.1) follows from Lemma 3.2. By assumption (1),

$$N\left(W(x)\left[V_i, Y_j\right]\right) \leqslant n(x) \tag{3.2}$$

for all $x \in \mathbb{R}^t$. If $W(x)[V_i, Y_j]$ is identically zero, we may improve (3.2) to $N(W(x)[V_i, Y_j]) = 0$. If $W(x)[V_i, Y_j] \neq 0$, then by assumption, all the entries of x appearing in $W(x)[V_i, Y_j]$ are indexed by a single set U_k. In this case, we may improve the inequality (3.2) to $N(W(x)[V_i, Y_j]) \leqslant n(x[U_k])$. Thus, the definitions of $\hat{W}$ and $\hat{x}$ give the entrywise inequality

$$0 \leqslant \left[N\left(W(x)\left[V_i, Y_j\right]\right)\right] \leqslant \hat{W}(\hat{x}).$$

Thus, the second inequality in (3.1) follows from Lemma 3.1. The third inequality in (3.1) is assumption (2), and the equality at the end of (3.1) is a property of the Euclidean norm that may be verified by direct calculation. ■

4. FURTHER REMARKS; FORBIDDEN SUBPATTERN NECESSARY CONDITIONS

If W is a $k \times k$ sprinkling that is not extendable but satisfies the inequality (0.1), one may construct examples that demonstrate that Corollary 2.7 does not supply a necessary condition for (0.1).

EXAMPLE 4.1. Let $W : \mathbb{R}^t \to M_{r,d}$ be a sprinkling that satisfies (0.1) and is not extendable. Then

$$\tilde{W} : \mathbb{R}^t \to M_{3r,3d}$$

defined by

$$\tilde{W}(x) = \begin{bmatrix} W(x) & 0 & 0 \\ 0 & W''(x) & 0 \\ 0 & 0 & W^T(x) \end{bmatrix}$$

satisfies (0.1), since

$$N\left(\tilde{W}(x)\right) = \max\left\{N(W(x)), N(W''(x)), N\left(W^T(x)\right)\right\}$$

for all $x \in \mathbb{R}^t$. The sufficient condition of Corollary 2.7 is not met by $\tilde{W}$, since the 1,1 block is not extendable, the 2,2 block fails condition (3) of Proposition 2.6, and the 3,3 block of $\tilde{W}^T$ is not extendable.

Another construction of this type is given in the next example:

EXAMPLE 4.2. Let $W : \mathbb{R}^t \to M_{r,d}$ be a sprinkling that satisfies the inequality (0.1) but is not extendable. Define $\tilde{W} : \mathbb{R}^{3t} \to M_{2r,2d}$ by the following: Let

$$
\begin{aligned}
U_1 &= \{1,\dots,t\},\\
U_2 &= \{t+1,\dots,2t\},\\
U_3 &= \{2t+1,\dots,3t\},\\
V_1 &= \{1,\dots,r\},\\
V_2 &= \{r+1,\dots,2r\},\\
Y_1 &= \{1,\dots,d\},\\
Y_2 &= \{d+1,\dots,2d\}.
\end{aligned}
$$

Then set

$$
\tilde{W}(x) = \begin{bmatrix} W(x[U_1]) & 0 \\ W^T(x[U_2]) & W''(x[U_3]) \end{bmatrix}.
$$

This sprinkling satisfies (0.1) by application of Proposition 3.3 with the U, V, and Y partitions indicated above. However, this sprinkling does not satisfy the sufficient condition of Corollary 2.7.

EXAMPLE 4.3. Here, we use the sprinkling of (2.1) to construct a sprinkling $\tilde{W}:\mathbb{R}^{12} \to M_{13,13}$ that satisfies the inequality (0.1) but again does not satisfy the condition of Corollary 2.7. Denoting the entries of $x \in \mathbb{R}^{12}$ by $a, b, c, \dots, k, l$, define $\tilde{W}(x)$ by Table 1. Note that W of (2.1) appears as the 5-by-5 submatrix in the upper left corner, W^T appears as the 5-by-5

TABLE 1

e	a	0	0	0	0	0	0	0	0	0	0	0
0	d	e	0	0	0	0	0	0	0	0	0	0
0	0	c	d	0	0	0	0	0	0	0	0	0
0	0	0	b	c	0	0	0	0	0	0	0	0
b	0	0	0	a	0	0	0	0	0	0	0	0
0	0	0	0	h	0	0	l	0	0	0	0	0
0	0	0	h	0	i	0	0	0	0	0	0	0
0	0	0	0	i	0	j	0	0	0	0	0	0
0	0	0	0	0	j	0	k	c	0	0	0	e
0	0	0	l	0	0	k	0	f	g	0	0	0
0	0	0	0	0	0	0	0	0	c	b	0	0
0	0	0	0	0	0	0	0	0	0	g	e	0
0	0	0	0	0	0	0	0	0	0	0	b	f

submatrix in the lower right corner, and W'' appears as the submatrix in rows indexed by $\{6,7,8,9,10\}$ and columns indexed by $\{4,5,6,7,8\}$. To demonstrate that $\tilde{W}$ satisfies (0.1), we use the following idea:

If $B \in M_{r,d}$ may be partitioned

$$B = [B_1, B_2, \dots, B_k],$$

where $B_i \in M_{r,d_i}$, $i = 1,\dots,k$, $d_1 + \dots + d_k = d$, and $B_i^T B_j = 0$ whenever $i \neq j$, then

$$N(B) = \max\{N(B_1),\dots,N(B_k)\}.$$

We add rows to $\tilde{W}$ to obtain a signed sprinkling

$$V = [V_1, V_2, \dots, V_9],$$

$V_1 \in M_{31,5}$, $V_i \in M_{31,1}$ ($i = 2,\dots,9$), with $V_i^T V_j = 0$ for $i \neq j$. The last 18 rows of V are as shown in Table 2. The inequality

$$N(V_i(x)) \leqslant n(x), \qquad 2 \leqslant i \leqslant 9,$$

TABLE 2

0	0	0	i	0	$-h$	0	0	0	0	0	0	0
0	0	0	k	0	0	$-l$	0	0	0	0	0	0
0	0	0	f	0	0	0	0	$-l$	0	0	0	0
0	0	0	g	0	0	0	0	0	$-l$	0	0	0
0	0	0	0	l	0	0	$-h$	0	0	0	0	0
0	0	0	0	j	0	$-i$	0	0	0	0	0	0
0	0	0	0	0	k	0	$-j$	0	0	0	0	0
0	0	0	0	0	c	0	0	$-j$	0	0	0	0
0	0	0	0	0	e	0	0	0	0	0	0	$-j$
0	0	0	0	0	0	c	f	$-k$	0	0	0	0
0	0	0	0	0	0	f	$-c$	0	0	0	0	0
0	0	0	0	0	0	g	0	$-k$	0	0	0	0
0	0	0	0	0	0	0	e	0	0	0	0	$-k$
0	0	0	0	0	0	0	0	e	0	0	0	$-c$
0	0	0	0	0	0	0	0	g	$-f$	0	0	0
0	0	0	0	0	0	0	0	0	b	$-c$	0	0
0	0	0	0	0	0	0	0	0	0	e	$-g$	0
0	0	0	0	0	0	0	0	0	0	0	f	$-b$

holds trivially for all x. The inequality

$$N(V_1(x)) \leqslant n(x)$$

holds because V_1^T is a sprinkling that satisfies the criteria of Proposition 2.3.

We note two further examples of sprinklings that satisfy the inequality (0.1) (since their respective transposes satisfy the criteria of Proposition 2.3) but are not extendable:

$$\begin{bmatrix} a & d & 0 & 0 \\ 0 & b & a & 0 \\ 0 & 0 & c & b \\ c & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} a & d & 0 & 0 & 0 \\ 0 & b & a & 0 & 0 \\ 0 & 0 & c & e & 0 \\ 0 & 0 & 0 & b & c \\ e & 0 & 0 & 0 & d \end{bmatrix}.$$

The following examples are sprinklings that do *not* satisfy (0.1):

$$W_1(a) = [a \quad a], \tag{4.1}$$

$$W_2\begin{pmatrix} a \\ b \end{pmatrix} = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix}, \tag{4.2}$$

$$W_3\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{bmatrix} a & c & 0 \\ b & 0 & c \\ 0 & b & 0 \end{bmatrix}. \tag{4.3}$$

Let $W, \tilde{W} : \mathbb{R}^t \to M_{r,d}$ be sprinklings. We will say that W and $\tilde{W}$ are permutation equivalent provided

$$\tilde{W}(x) = PW(Rx)Q$$

for some permutation matrices P, R, and Q. Clearly, W satisfies the inequality (0.1) if and only if $\tilde{W}$ does. W_3 is permutation equivalent to its transpose, and also to W_3', W_3'', and their transposes. Likewise with W_2. With W_1, the possibilities of W_1,

$$W_1^T(a) = \begin{bmatrix} a \\ a \end{bmatrix}, \tag{4.4}$$

and

$$W_1'\begin{pmatrix} a \\ b \end{pmatrix} = [a+b]. \tag{4.5}$$

We note that W_1' is not a sprinkling.

We will say the sprinkling W *contains* $\tilde{W}$ if $\tilde{W}$ can be obtained from W by

(1) replacing nonzero entries with zero,
(2) deleting rows and/or columns.

The following lemma gives the significance of the *contains* relation.

LEMMA 4.4. *If W contains $\tilde{W}$ and W satisfies the inequality* (0.1), *then $\tilde{W}$ satisfies* (0.1).

Proof. If $\tilde{W}$ is obtained from W by deleting only entire rows and/or columns, then $\tilde{W}$ is a submatrix of W, so $N(\tilde{W}(x)) \leqslant N(W(x))$ holds for all x. If $\tilde{W}$ is obtained from W by replacing nonzero entries with zero, then for any nonnegative vector v, the inequalities $0 \leqslant \tilde{W}(v) \leqslant W(v)$ hold entrywise, so by Lemma 3.1, $N(\tilde{W}(v)) \leqslant N(W(v))$ holds. Next, we argue that if W satisfies $N(W(v)) \leqslant n(v)$ for all nonnegative vectors v, then the inequality holds for all vectors. Let x be given. Then

$$\begin{aligned} N(W(x)) &\leqslant N(|W(x)|) = N(W(|x|)) \\ &\leqslant n(|x|) = n(x), \end{aligned}$$

where the first inequality follows from Lemma 3.2 with $V_i = \{i\}$ and $Y_j = \{j\}$. ■

Our necessary condition for (0.1) is the following immediate corollary:

COROLLARY 4.5. *If the sprinkling W satisfies the inequality* (0.1), *then it contains no sprinkling permutation equivalent to W_1, W_1^T, W_2, or W_3 (defined in* (4.1), (4.2), (4.3), *and* (4.4)).

We would like to propose:

PROBLEM 4.6. Is the lack of a forbidden subpattern identified in Proposition 4.5 a sufficient condition for the inequality (0.1)?

5. THE CASES OF SPRINKLINGS WITH THREE OR FOUR VARIABLES

In this section, we show that for sprinklings $W : \mathbb{R}^3 \to M_{r,d}$, the answer to the question of Problem 4.6 is yes. We also show that a sprinkling $W : \mathbb{R}^4 \to M_{r,d}$ that does not contain one of the forbidden patterns (4.1), (4.2), (4.3), or (4.4) satisfies a weaker, combinatorial inequality.

PROPOSITION 5.1. *A sprinkling $W : \mathbb{R}^3 \to M_{r,d}$ satisfies the inequality* (0.1) *if it does not contain a sprinkling permutation equivalent to one of the forms given in* (4.1), (4.2), (4.3), *or* (4.4).

Proof. We show that if W contains only three variables and does not contain one of the four forbidden patterns, then it is permutation equivalent to a direct sum of blocks of the form

$$\begin{bmatrix} a & b & c \end{bmatrix}, \quad \begin{bmatrix} a & b & 0 \\ c & 0 & b \end{bmatrix}, \quad \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}, \quad \begin{bmatrix} a & b \end{bmatrix} \text{ or } \begin{bmatrix} a \end{bmatrix}$$

and their transposes.

We argue as follows: Since W does not contain W_1, each row contains at most three nonzero entries. Since W does not contain W_1^T or W_2, any row with three nonzeros must be combinatorially orthogonal to every other row. Hence, it is possible to reorder the rows and columns of W to obtain

$$\begin{bmatrix} a & b & c \end{bmatrix} \oplus \dots \oplus \begin{bmatrix} a & b & c \end{bmatrix} \oplus \hat{W}_1$$

in which the block $\hat{W}_1$ has at most two nonzero entries in each row. Repeating this argument with columns instead of rows, we may obtain

$$\begin{bmatrix} a & b & c \end{bmatrix} \oplus \dots \oplus \begin{bmatrix} a & b & c \end{bmatrix} \oplus \begin{bmatrix} a \\ b \\ c \end{bmatrix} \oplus \dots \oplus \begin{bmatrix} a \\ b \\ c \end{bmatrix} \oplus \hat{W}_2,$$

in which the block $\hat{W}_2$ contains at most two nonzeros in each row and each column. Now consider a row of $\hat{W}_2$ with two nonzeros. By permutation of rows and columns we may move that row to the top of the matrix and put the nonzeros at the left:

$$\hat{W}_2 = \begin{bmatrix} a \quad b & 0 \\ ?_{2,1} & ?_{2,2} \end{bmatrix}$$

We consider three possibilities for the $?_{2,1}$ block:

Case 1. If the $?_{2,1}$ block is all zeros, we have

$$\hat{W}_2 = \begin{bmatrix} a \quad b & 0 \\ 0 & \hat{W}_3 \end{bmatrix}$$

Case 2. If the $?_{2,1}$ block contains two nonzeros, then by permuting rows we may obtain

$$\hat{W}_2 = \begin{bmatrix} \begin{matrix} a & b \\ c & 0 \\ 0 & c \end{matrix} & 0 \\ 0 & \hat{W}_3 \end{bmatrix}.$$

The 1,2 block must be zero, since placing a nonzero anywhere in that block produces one of the forbidden patterns.

Case 3. If the $?_{2,1}$ block contains one nonzero, then by permuting rows and columns we may obtain

$$\hat{W}_2 = \begin{bmatrix} \begin{matrix} a & b \\ c & 0 \end{matrix} & 0 \\ 0 & ?_{2,2} \end{bmatrix}.$$

We consider two subcases, the first is when row two of $\hat{W}_2$ has one nonzero and the second is when row two of $\hat{W}_2$ has two nonzeros.

Case 3a. Row 2 of $\hat{W}_2$ has one nonzero. Then

$$\hat{W}_2 = \begin{bmatrix} \begin{matrix} a & b \\ c & 0 \end{matrix} & 0 \\ 0 & \hat{W}_3 \end{bmatrix}.$$

Case 3b. Row 2 of $\hat{W}_2$ has two nonzeros. Then by permuting the columns of $\hat{W}_2$ we may obtain

$$\hat{W}_2 = \begin{bmatrix} a & b & 0 & \\ c & 0 & c & 0 \\ & 0 & & \hat{W}_3 \end{bmatrix}.$$

The 1,2 block in cases 3a and 3b must be zero to avoid the forbidden patterns. In case 3b, the 2,1 block must be zero, since placing a nonzero anywhere in this block produces too many nonzeros or a forbidden subpattern.

Thus, we may permute the rows and columns of W to obtain

$$W = \tilde{W}_1 \oplus \tilde{W}_2 \oplus \ldots \oplus \tilde{W}_k \oplus \tilde{W},$$

where each $\tilde{W}_i$, $i = 1, \ldots, k$, is of the form

$$[a \quad b], \qquad \begin{bmatrix} a & b \\ c & 0 \\ 0 & c \end{bmatrix}, \qquad \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} a & b & 0 \\ c & 0 & b \end{bmatrix}.$$

Now, $\tilde{W}$ has at most one nonzero in each row and at most two nonzeros in each column. Thus, by permuting rows and columns, $\tilde{W}$ may be put into the form

$$\tilde{W} = \tilde{W}_{k+1} \oplus \ldots \oplus \tilde{W}_q \oplus 0,$$

in which the zero summand may or may not be present and $\tilde{W}_i$, $i = k + 1, \ldots, r$, is of the form

$$\begin{bmatrix} a \\ b \end{bmatrix} \quad \text{or} \quad [a]. \qquad \square$$

We can prove a weaker inequality for a sprinkling $W : \mathbb{R}^4 \to M_{r,d}$.

PROPOSITION 5.2. *The number of nonzeros in a four-variable sprinkling that does not contain a permutation equivalence of the sprinklings of* (4.1), (4.2), (4.3), *or* (4.4) *is at most* $r + d$.

When $r = d$, this is equivalent to the inequality (1.3), weakened by requiring x, y, and z to be vectors of all ones.

Proof. Suppose that W is r-by-d, has more than $r + d$ nonzero entries, and is minimal with respect to $r + d$. Minimality implies that every row and column must have at least two nonzeros. If W has a row with more than two nonzeros, then by permuting rows and columns,

$$W = \left[\begin{array}{ccc|c} a & b & c & \\ d & 0 & 0 & \\ 0 & d & 0 & W_{12} \\ 0 & 0 & d & \\ & 0 & & W_{22} \end{array}\right].$$

However, then $W_{12} \neq 0$ provides a forbidden pattern; hence there are rows with one nonzero entry, a contradiction. Hence, our minimal W has exactly two nonzero entries in each row. The same argument applies to columns, so that the total number of nonzero entries is $2r = 2d$. ■

This suggests the purely combinatorial

PROBLEM 5.3. Prove to find a counterexample: If $W : \mathbb{R}^t \to M_{r,d}$ is a sprinkling and does not contain a sprinkling permutation equivalent to the sprinklings of (4.1), (4.2), (4.3), or (4.4), then the number of nonzero entries in W does not exceed $\sqrt{rdt}$.

This amounts to asking if the inequality (1.3) holds with each of the vectors x, y, and z in (1.3) having each entry equal to one. Thus it is a weakening of (1.3)—equivalently, a weakening of (0.1).

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